On the stability of stochastic dynamic systems and their use in econometrics

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Chapter 1

Introduction

1.1 Time-series literature: a selective review

A time series is a collection of observations that are indexed in the order of time. Such data structures are found in many fields of science (such as economics, finance, demographics, etc) and business (such as risk assessment, inventory management, logistics optimization, etc). The main motivation for studying time series is the assumption that the past and the future behaviour of a process contain similarities, and thus that one can build predictions for the future based on past observations. The connection between past and future is quite a delicate one, if the dependence is very weak then prediction is fruitless, while a too complex dependence makes prediction very complicated. Statistical analysis and modelling tries to describe the dependence by designing a collection of possible probabilistic descriptions of the data, called a model, and then using the observed past to decide on the most appropriate specification within the model through the construction of an estimator.

The popularisation of time series models started with linear models such as the autoregressive moving average model (ARMA) of Box et al. (1970). The fundamental idea was to let dependence change over time and instead fix the conditional dependence. That is, given yesterday we expect different dependence for today and tomorrow, but the way today depends on yesterday and tomorrow depends on today is the same. The linearity of the models makes it possible to get closed forms for most entities of interest and thus makes the analysis of the models very tractable. ARMA models are able to describe a large class of time series dynamics, in fact, the renowned Wold decomposition theorem

(Wold, 1938) shows that any covariance-stationary time series has a representation as the sum of a deterministic time series and an infinite moving average. Therefore such time series can theoretically be approximated by taking a large number of lags in the ARMA specification.

Nevertheless, a statistical model is only as strong as the closest description to the actual data generating process that it contains. Therefore statisticians have proceeded by enlarging models to increase the likeliness of containing such a close description. Nonlinear statistical models have become increasingly popular with models such as threshold models (Tong and Lim, 1980; Zakoian, 1994) or Markov regime switching models (Hamilton, 1989) that allow different ranges of the state space to have distinct effects on the dependence structure. A general way to extend a given model to a larger collection of distributions is to make it dynamic by choosing a parameter and making it time-varying. The most famous application of this approach can be found in heteroskedastic volatility models. The pioneering examples of such models are the ARCH model of (Engle, 1982), that has led to a large portion of literature on extended specifications such as the GARCH (Bollerslev, 1986) and the EGARCH (Nelson, 1991) model, and the stochastic volatility model of Taylor (2008) that has lead to many extensions both specification (Harvey and Shephard, 1996) and estimation (Jacquier et al., 2002) wise.

There are two general classes of time-varying parameter models. The first class, called parameter-driven, specifies the dynamics of the time-varying parameter as a new stochastic data generating process with its own disturbance process. The second class, denoted observation-driven, describes the time varying parameter at each point of time as a function of a, potentially infinitely long, sequence of past observations of the data. Both strands of time-varying parameter time series models have their advantages and disadvantages. Parameter driven models typically satisfy desired statistical stability properties, however the likelihood usually does not have a closed form so that computationally intensive simulation techniques are needed to find the best fitting specification within the model. Observation driven models do have a closed form for the likelihood and thus are generally easier to estimate. However, showing that the model is statistically stable typically requires careful mathematical analysis.

Stability features of both the data generating process behind the observations and the unobserved time varying parameter are very helpful. Statistical properties such as stationarity, ergodicity and mixing provide the basis for the analysis of limiting estimator behavior as the amount of observations goes to infinity. Specifically they imply versions of the law of large numbers and the central limit theorem that can then be applied to show consistency (convergence to the true unknown model) and asymptotic normality (convergence rate and asymptotic distribution). The stability properties for linear models can be fully characterised using lyapunov exponents as is done in Bougerol and Picard (1992a,b). Nonlinear models require a lot more work and have traditionally been studied using Markov chain theory to obtain geometric ergodicity as in (Meyn and Tweedie, 2012). Verifying the underlying assumptions regarding proper commuting behavior of the Markov chain over the state space can be difficult, but once those are satisfied, ergodicity follows very generally from moderate "Foster-Lyapunov" drift criteria that essentially ensure that the Markov chain never wanders too far off.

Time-varying parameter models pose a new challenge, because the parameter process itself is an unobserved component of the model. As a solution statisticians construct an approximation for the process by choosing a starting point and then recursively filtering the time-varying parameter. Deriving limiting behaviour of this process then requires that the approximation converges to a stable solution, a property that has been denoted invertibility in Straumann (2005). Bougerol (1993) and Straumann (2005) propose a method to obtain invertibility that is based on stochastic recurrence equations (SREs) satisfying contraction conditions. Their result is a stochastic variant of Banach's fixed point theorem and requires the stochastic recurrence equation (SRE) to be uniformly contracting over the whole state space, on average. Although the condition is fairly elementary to write down for a given model, it introduces some complications. The one step contraction condition is typically possible to derive analytically, but imposes a very strict restriction on the possible parameter values within the model. Typically, SREs exhibit non contractive areas over small parts of the state space, which by the uniform condition imply that the stability parameter region becomes impractically small. Resorting to higher fold contraction conditions leads to larger stability regions, however these regions cannot be analytically determined as they depend on the unknown data generating process. Alternatively, one can compute empirical stability regions as in Wintenberger (2013) and Blasques et al. (2018a).

1.2 Contributions of the thesis

This thesis contains three chapters of research that explore the stability of time series models in a purely theoretic, a financial and a macroeconomic setting. Chapter 2 and Chapter 3 are somewhat linked and explore invertibility for a class of observation driven time-varying parameter models. Chapter 4 is fully self contained and discusses nonlinear macro economic time series models without time-varying parameters. I provide a short description of each chapter below, a more detailed explanation including relevant references in the literature can be found in the introduction of each specific chapter.

Chapter 2 is joint work with Francisco Blasques and discusses a novel invertibility condition that provides a stability region for a large collection of models that is typically impossible to analyze using the contraction condition of Bougerol (1993). Specifically the condition allows for discontinuities and explosive, non-contracting or chaotic behavior of the SRE over parts of the state space. In return for relaxing the contraction condition we impose that the SRE satisfies a resetting requirement that involves there being a positive probability of the SRE updating to a value that is independent of the past of the process. The proof that this implies invertibility can be summarised in one sentence: it is not important how far two paths diverge from one another as long as they eventually collapse to the same value. That means that between any two resetting times, any imaginable sample path behavior is allowed as the reset ensures that it returns to a fixed value. The resetting condition seems strong, but is typically satisfied in time series that exhibit bubble collapses, in which case the collapse itself can be chosen as the resetting moment. Many time series contain these collapsing dynamics such as volatility bubbles studied in Saïdi (2003) and Saïdi and Zakoian (2006), financial bubbles studied in Gouriéroux and Zakoïan (2013, 2017), Blasques et al. (2018b) or Chapter 3 of this thesis, and time series for overshooting predator-prey populations such as the famous Canadian lynx-hare and wolf-moose datasets studied in Tsay (1989) or Teräsvirta (1994). Additionally, the framework lends itself naturally to regime switching models, where we then make one regime independent of the past so that it enforces a reset and in return get complete freedom for the other regimes. We illustrate the generality of the theory and how to apply it by deriving the specific parameter region for the volatility bubble model in Saïdi and Zakoian (2006).

Chapter 3 is joint work with Francisco Blasques and Siem Jan Koopman and studies speculative bubbles in time series of financial asset prices. The rational expectations literature on asset pricing models the asset price process as the sum of a fundamental value process and a locally explosive process that describes a burst followed by sharp mean-reverting dynamics. We mimic this approach within an observation driven time-varying parameter model, where we split the level in a time varying fundamental value process and a bubble specification. We provide a general framework that encompasses a wide possibility of bubble dynamics as the interplay between the fundamental value and bubble process allows for various impulse response functions. We estimate the model using a classical maximum likelihood approach. All the stability properties such as stationarity, ergodicity, mixing and invertibility are proven using the theoretical results from Chapter 2. We illustrate the flexibility of the model by filtering the bitcoin / US dollar exchange rate around its biggest relative bubble and show that in sample predictions anticipate the bubble collapse in advance. We give some insights into the advantages of observation driven models and the ease at which they allow for the derivation of quantities such as bubble burst probabilities, bubble emergence probabilities or expected bubble life times.

Chapter 4 is joint work with Francisco Blasques and provides econometric foundations for perturbation, a well known method to approximate solutions of dynamic stochastic equilibrium models. Many approximation methods exist and there are some properties to keep in mind when selecting one, mainly: accuracy, speed, stability and accessability. Arbitrarily accurate global approximation methods exist such as value function iteration (Bertsekas, 1987), projection (Judd, 1992) or machine learning Norets (2012). Typically these methods are less accessible to the practitioner as they are more complex to code. Moreover, traditionally these methods were too slow for an estimation setting and thus require additional techniques such as parallel computing and entering parameters as pseudo-states in the model. Perturbation (Judd and Guu, 1997; Schmitt-Grohé and Uribe, 2004) is the most commonly used method and focusses on speed and accessability by approximating the solution via a Taylor expansion around the deterministic steady state. It is well known, however, that a higher-order polynomial, and thus perturbation, defines an unstable dynamic system which produces explosive paths. This means that none of the typical stability properties hold and thus that existence of relevant moments and consistency or asymptotic normality of estimators cannot be derived. In order to deal with the unstable dynamics of higher-order perturbation solutions, Kim et al. (2008) proposed the pruning method. The pruning method has been successfully implemented in software packages and effectively solves the problem of explosive dynamics. New complications are introduced however, pruning is a simulation-based approximation and hence does not provide a policy function. Moreover, the method has to sacrifice local accuracy of the approximation to obtain stability. Our paper introduces a simple correction to perturbation solutions that is designed to enrich perturbation solutions with all the desirable stochastic properties needed for parameter estimation and statistical inference. Our correction transforms the standard perturbation approximation by replacing higher order monomials in the Taylor expansion with transformed ones that are based on the transformed polynomials introduced in Blasques et al. (2014). These transformed monomials force sample paths that move far away from the deterministic steady state into linear dynamics, which makes the resulting dynamic system a prime candidate to be analysed within a Markov chain setting. We prove that transformed perturbation produces non explosive paths and that solutions are stationary and ergodic with bounded moments to which sample moments of the process converge. Finally we demonstrate that our method is very accurate within the setting of fast and accessible solution methods. We provide a detailed analysis and comparison with both first order perturbation and pruning for two nonlinear DSGE models in which higher order perturbation is infeasible.